Hybrid backscattering model by a prolate spheroid

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Created on January 5, 2024

Revised on March 26, 2024

The resonance scattering by a prolate spheroid (Ye, 1997):

$$f_{bs_0}^{res} = \frac{a_{esr}F_{Spheroid}^2}{\left(\frac{\omega_{0_{esr}}^2}{\omega^2}\right)F_{spheroid}-1-ika_{esr}F_{spheroid}^2} \tag{1}$$

where

$$\omega_{0_{esr}} = \frac{2\pi}{a_{esr}}\sqrt{\frac{3\gamma P_e}{\rho_w}} \tag{2a}$$

$$\omega_{spheroid} = \omega_{0_{esr}}F_{spheroid}^{1/2}, \tag{2b}$$

$$F_{spheroid}^{1/2} = 2^{\frac{1}{2}}e_{ac}^{-\frac{1}{3}}(1-e_{ac}^2)^{\frac{1}{4}}\left[\ln\left(\frac{1+\sqrt{1-e_{ac}^2}}{1-\sqrt{1-e_{ac}^2}}\right)\right]^{-\frac{1}{2}} \tag{2c}$$

where $e_{ac} = \frac{a}{c} < 1$ is the aspect ratio, the ratio of semi-minor to semi-major axis, and $e_{ac} = 1$ for the case of a sphere. It is easy to verify that $F_{spheroid} = 1$ for $e_{ac} = 1$, i.e., for the case of a sphere.

The corresponding differential backscattering cross section is then

$$\sigma_{bs_0}^{res}(ka_{esr}) = \left|f_{bs_0}^{res}\right|^2 = \frac{a_{esr}^2F_{spheroid}^2}{\left[\left(\frac{\omega_{0_{esr}}}{\omega}\right)^2F_{spheroid}^2-1\right]^2+k^2a_{esr}^2F_{spheroid}^2}, \tag{3}$$

To add the angle of orientation dependent term, we use a Sinc function

$$\Theta(\theta) = \frac{\sin(kc\sin\theta)}{kc\sin\theta}, \tag{4}$$

where θ is the angle of orientation of the prolate spheroid (normal of the major axis) relative to the incident wave. The combined scattering angle-dependent scattering model is then

$$\sigma_{bs}^{res}(ka_{esr},\theta) = \Theta(\theta)\sigma_{bs_0}^{res}(ka_{esr}) \tag{5}$$

For $ka_{esr} \gg 1,$ or as $ka_{esr} \to \infty$, from the Kirchoff Approximation when frequency is very high, Eq. (64) of Gaunuard (1985) and for $r, R \gg a_1, a_2 > \lambda$

$$\sigma_{bs}^{asymp}(\theta) = \frac{a_1(\theta)a_2(\theta)}{4} \quad (6)$$

It can be derived that for an ellipsoid with incident wave in the X-Z plane,

$$\sigma_{bs}^{asymp}(\theta) = \frac{\sigma_{bs_0} e_{ac}^4}{\left(\sin^2\theta + e_{ac}^2 \cos^2\theta\right)^2}, \quad (7)$$

where $\sigma_{bs_0} = \sigma_{bs}(0)$ is the broadside incidence:

$$\sigma_{bs_0} = \frac{c^2}{2a}\left(\frac{b^2}{2a}\right) = \frac{c^2}{4} e_{ba}^2, \quad (8)$$

where $e_{ba} = \frac{b}{a}$ and $e_{ba} = 1$ for a prolate spheroid.

We construct a following transition function (a transient response of a passive electric system to a step function, Alexander et al., 2013):

$$\sigma_{bs}^{hybrid}(ka_{esr}, \theta) = \left[\sigma_{bs}^{asymp}(\theta) - \sigma_{bs}^{res}(k_T a_{esr}, \theta)\right]\left(1 - e^{-\alpha(ka_{esr} - k_T a_{esr})}\right) + \sigma_{bs}^{res}(k_T a_{esr}, \theta), \quad (9)$$

where k_T is the wave number corresponding to the frequency where the transition starts, α is a parameter governing the transition rate.

$$TS^{hybrid} = 10 \log_{10} \sigma_{bs}^{hybrid}(ka_{esr}, \theta) \quad (10)$$

Comparison of this analytical model with the numerical simulations using the Boundary Element Method (BEM) is shown in Fig. 1, where we plot the reduced target strength (*RTS*) as a function of ka_{esr}.

$$RTS^{hybrid} = 10 \log_{10}\left(\frac{\sigma_{bs}^{hybrid}(ka_{esr}, \theta)}{\sigma_{bs_0}}\right). \quad (11)$$

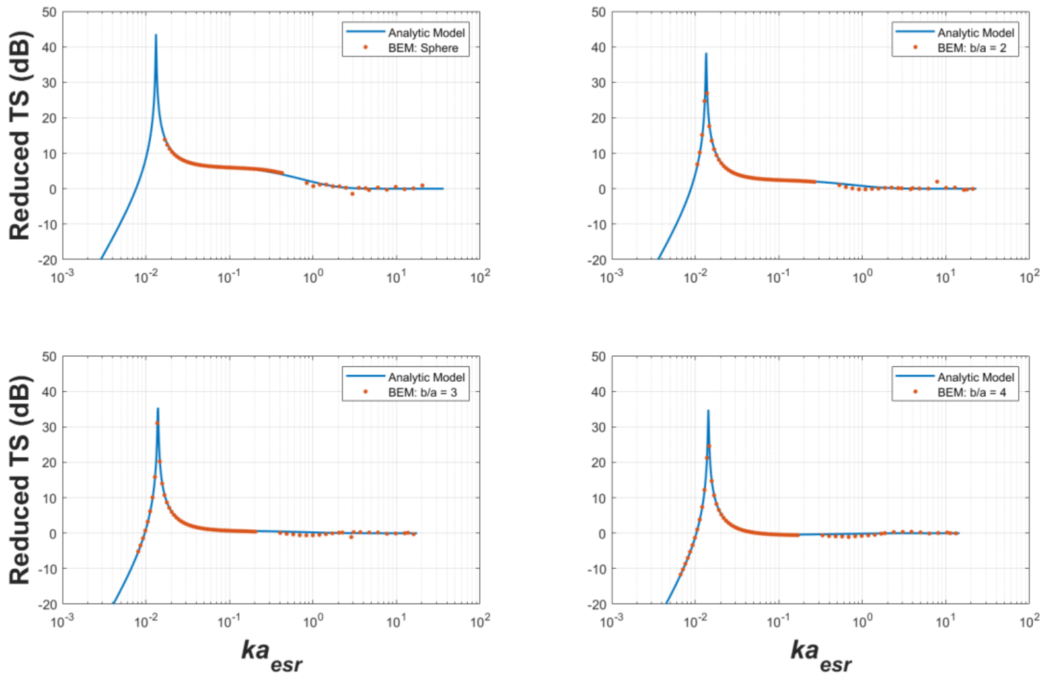

Figure 1. Comparison between the analytical model for prolate spheroids (Eq. 9) with the BEM simulations for aspect ratios of 1 (sphere), 2, 3, and 4, respectively, where the y-axis is the Reduced Target Strength, and x-axis is the dimensionless quantity ka_{esr}. In Eq. (9), a transition parameter $k_T a_{esr} = 0.1$ and $\alpha = 1.3$ are used in the hybrid model.

References

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Ye, Z. 1997. Low-frequency acoustic scattering by gas-filled prolate spheroids in liquids. *J. Acoust. Soc. Am.*, 101: 1945-1952. https://doi.org/10.1121/1.418225